ANALYTIC CONTINUATION BY SUMMATION-METHODS

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ABSTRACT

The paper deals with the analytic continuation of the geometric series by a family of linear transformations into some special domains of the complex plane.

1. **Introduction.** The problem of analytic continuation by summability may be formulated as follows: Let $f(z)$ have the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \tag{1.1}$$

with a positive radius of convergence. Two questions arise: (i) What is the domain of efficiency of a special linear transformation of (1.1) regarding the analytic continuation of $f(z)$? (ii) Given some domain in the complex plane, does there exist a linear transformation of (1.1) which yields the analytic continuation of $f(z)$ exactly into this domain and nowhere else?

In some cases, as has been shown by Borel [1], Okada [4] and Vermes [7], it is sufficient to focus attention on the continuation of the geometric series $\sum z^n$, $|z| < 1$; in this paper we deal only with the above series. In this context, Dienes and Cooke [2] have shown that there exist transformations that are effective at some distinct points outside the circle of convergence; this result was extended by Vermes [8] to a denumerable set of points. Russel [5] and Teghem [6] have produced transformations effective, respectively, on Jordan arcs and on domains that are not simply-connected.

Definitions and Notations. Corresponding to a real or complex sequence $\{d_k\}$, $(d_k \neq -1)$, the generalized Lototski or $[F, d_n]$-transform $\{t_n\}$ of a sequence $\{s_n\}$ is defined by Jakimovski [3]:

$$t_n = \prod_{k=1}^{n} (d_k + 1)^{-1}(d_k + E)(s_0), \qquad n \geqq 1 \tag{1.2}$$

where

$$E^p(s_k) = s_{p+k} \qquad k \geqq 0, \ p \geqq 0.$$

If $\lim t_n$ exists as $n \to \infty$, we say that $\{s_n\}$ is summable $[F, d_n]$ to the value $\lim t_n$.

We shall also use the following method of summation: For every sequence of polynomials $\{P_n(x)\}$ satisfying $P_n(1) \neq 0$, the $[F^*, P_n]$-transform of a sequence $\{s_n\}$ will be defined by

$$t_n^* = \prod_{k=1}^{n} (P_k(1))^{-1} P_k(E)(s_0), \qquad n \geqq 1. \tag{1.3}$$

It may easily be seen that if $\{s_n\}$ is the sequence of partial sums of the geometric series $\sum z^n$ $(z \neq 1)$, then in the notation above

$$t_n = (1-z)^{-1} - z(1-z)^{-1} \prod_{k=1}^{n} (d_k+1)^{-1} \cdot (d_k + z) \tag{1.4}$$

and

$$t_n^* = (1-z)^{-1} - z(1-z)^{-1} \prod_{k=1}^{n} (P_k(1))^{-1} \cdot P_k(z) \tag{1.5}$$

It follows that, for $z \neq 0, 1$, $\lim_{n\to\infty} t_n = (1-z)^{-1}$ if and only if

$$\lim_{n\to\infty} \prod_{k=1}^{n} (d_k+1)^{-1}(d_k+z) = 0, \tag{1.6}$$

while $\lim_{n\to\infty} t_n^* = (1-z)^{-1}$ if and only if

$$\lim_{n\to\infty} \prod_{k=1}^{n} (P_k(1))^{-1} P_k(z) = 0. \tag{1.7}$$

2. The main results.

THEOREM 1. *Let the polynomial $P(z)$ satisfy*

$$Re\,P(1) = 0\ . \tag{2.1}$$

Then, there exists a fixed sequence $\{d_n\}(n \geqq 1)(d_n \neq -1)$ such that the $[F, d_n]$-transform sums the geometric series to the value $(1-z)^{-1}$ for every z for which $Re\,P(z) > 0$, and does not sum it for every z for which $Re\,P(z) < 0$. The convergence of the transform is uniform in every bounded closed subset of $\{z; Re\,P(z) > 0\}$.

Proof.

Clearly we may suppose $P(z) \not\equiv$ const. Then for every $k \geqq 1$

$$P(z) + k = c(z + a_1^k)(z + a_2^k) \cdots (z + a_p^k) \tag{2.2}$$

where $p \geqq 1$, $c \neq 0$ and c does not depend on k. Define now $d_1 = a_1^1$, $d_2 = a_2^1, \cdots$, $d_p = a_p^1$, $d_{p+1} = a_1^2, \cdots$, $d_{2p} = a_p^2, \cdots$ and in general if $\nu = \mu p + \rho$ $(0 < \rho \leqq p)$

$$d_\nu = a_\rho^{\mu+1}. \tag{2.3}$$

Now let $n = mp + q$ $(0 \leqq q < p)$; then

$$\prod_{\nu=1}^{n} \frac{d_\nu + z}{d_\nu + 1} = \prod_{k=1}^{m} \frac{P(z)+k}{P(1)+k} \cdot \prod_{\nu=mp+1}^{mp+q} \frac{d_\nu + z}{d_\nu + 1} \equiv \prod\nolimits_1^{(n)} \cdot \prod\nolimits_2^{(n)} \tag{2.4}$$

where the second factor is 1 if $q = 0$. By (2.1), if $|1 - z| < \delta$ then $|Re P(z)| < \frac{1}{2}$, and by (2.2) and (2.3) for $1 \leqq \rho \leqq p$, $\mu \geqq 0$,

$$Re P(-d_{\mu p+\rho}) = -(\mu + 1) \leqq -1; \tag{2.5}$$

thus

$$|1 + d_\nu| \geqq \delta > 0 \qquad \nu = 1, 2, \cdots \tag{2.6}$$

$$\left|\prod\nolimits_2^{(n)}\right| = \left|\prod_{\nu=mp+1}^{mp+q} \left(1 + \frac{z-1}{d_\nu + 1}\right)\right| \leqq \prod_{\nu=mp+1}^{mp+q} \left(1 + \left|\frac{z-1}{d_\nu + 1}\right|\right), \tag{2.7}$$

and by (2.6) $\left|\prod_2^{(n)}\right| \leqq (1 + (|z - 1|)/\delta)^{p-1}$.

Thus $\prod_2^{(n)}$ is uniformly bounded for every $n \geqq 1$ and for every z belonging to a fixed bounded point-set.

$$\left|\prod\nolimits_1^{(n)}\right|^2 = \prod_{k=1}^{m} \left|\frac{P(z)+k}{P(1)+k}\right|^2 = \prod_{k=1}^{m} \left(1 + \frac{2k Re P(z) + |P(z)|^2 - |P(1)|^2}{k^2 + |P(1)|^2}\right) \tag{2.8}$$

By a well known theorem on infinite products

$$\lim_{n\to\infty} \prod\nolimits_1^{(n)} = \begin{cases} 0 & \text{if } Re P(z) < 0 \\ \infty & \text{if } Re P(z) > 0. \end{cases} \tag{2.9}$$

Also, the convergence to 0 is uniform in every point-set where $Re P(z) \leqq -\varepsilon$, with $\varepsilon > 0$ fixed. (2.9), (2.7), (2.4) and (1.6) prove the theorem.

EXAMPLE. (i) The Lototski-transform definited by $[F, d_n = n - 1]$ sums the geometric series for $Re z < 1$, and does not sum it for $Re z > 1$, [3]. Here $P(z) = z - 1$.

(ii) If $P(z) = e^{i\gamma}(z - 1)$ with a suitable real γ we obtain as domain of summability any given half plane, the boundary of which is a straight line passing through $z = 1$.

(iii) If $P(z) = e^{i\gamma}(z - 1)(z - \alpha - i\beta)$, with real α, β, γ, we obtain as domain of summability the "inside" or "outside" of hyperbolas passing through $z = 1$.

Next we prove the following theorem:

THEOREM 2. *Let R be a set that contains the point $z = 1$ and whose complement consists either of the point ∞ or of an unbounded domain. Let $f(z)$ be an analytic regular function on R satisfying*

(2.10) $$Re f(1) = 0 .$$

Then, there exists a sequence of polynomials $\{P_n(x)\}$ $(n \geqq 1, P_n(1) \neq 0)$ *such that the* $[F^*, P_n]$ *transformation sums the geometric series to the value* $(1 - z)^{-1}$ *for every* $z \in R$ *for which* $Re\ f(z) < 0$ *and does not sum it for* $z \in R$ *for which* $Re\ f(z) > 0$.

Proof. By the well-known theorem of Walsh [9] for every $k \geqq 1$ there exist polynomials $Q_k(z)$ satisfying

(2.11) $$|Q_k(z) - f(z)| < k^{-1}$$

for $z \in R$, $|z| \leqq k$, and

(2.12) $$Q_k(1) = f(1) \qquad k = 1, 2, \cdots$$

Define

(2.13) $$P_k(z) = Q_k(z) + k \qquad k = 1, 2, \cdots$$

By (2.11), (2.12) and (2.13) for any fixed z $(|z| \leqq k)$

$$(P_k(1))^{-1} \cdot P_k(z) = 1 + [f(z) - f(1)] \cdot k^{-1} + O(k^{-2}).$$

Now, by (2.10) and the theory of infinite products, if $z \in R$

$$\lim_{n \to \infty} \left| \prod_{k=1}^{n} (P_k(1))^{-1} P_k(z) \right| = \begin{cases} 0 & \text{for } Re\, f(z) < 0 \\ \infty & \text{for } Re\, f(z) > 0. \end{cases}$$

By (1.7) this proves the theorem.

REMARK. A generalization of Theorem 2 can be made to the situation where R is the union of an increasing sequence of bounded closed sets R_i the complement of each of which is an unbounded domain. This result will prove the existence of an $[F^*, P_n]$-transformation that is effective for $\sum z^n$ in the entire Mittag-Leffler star of $(1 - z)^{-1}$. It has to be mentioned that the $[F^*, P_n]$-transformations are row-finite. Because of the length of proof we only state the following result too:

THEOREM 3. *Let D be an union of a finite number of simply-connected bounded domains having Jordan boundaries. Let* $z = 1$ *lie on the boundary, and let E be a closed subset of the complement of D. Then there exists an* $[F^*, P_n]$*-transformation, which sums the geometric series to the sum* $(1 - z)^{-1}$ *for every* $z \in D$ *and does not sum it for every* $z \in E$.

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